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ABUNDANCE EFFECTS ON THE CEPHEID DISTANCE SCALE

RICHARD B. STOTHERS
NASA/Goddard Space Flight Center, Institute for Space Studies
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ABSTRACT

A new investigation is made of the dependence of the period-luminosity (PL) and period-luminosity-color (PLC) relations on the helium (Y) and metal (Z) abundances in classical Cepheids. For consistency in overall approach and in error estimation, theoretical data alone have been used. These data come from stellar evolution, atmosphere, and pulsation theory. To enhance the accuracy, relative quantities have been adopted to the exclusion of absolute quantities, and numerous published results for each quantity have been averaged before use. Most important of all, composition dependences have been assigned to *all* quantities. The results therefore represent a true theoretical prediction.

A major quantitative result is that the PLC relation in B, V magnitudes is at least 5 times more sensitive to chemical composition than is the PL relation, whose dependence is $\delta M_V(\mathrm{PL}) \approx 0.5\delta Y - 2.8\delta Z$ at a fixed period. If the PL relation is adopted and the helium abundance is held fixed, a Cepheid with $Z = 0.005$ should be fainter than one with $Z = 0.02$ by ~ 0.04 mag, a result which falls between Iben and Renzini's value of zero and Gascoigne's value of 0.1–0.2 mag but is small in any case. Since the slope of the predicted PL relation is independent of chemical composition, the PL relation can be considered to be effectively universal. Supporting observational evidence consists of (1) the near invariance of the visual absolute magnitude difference between Cepheids with 6 day period and RR Lyrae stars in the Small Magellanic Cloud, Large Magellanic Cloud, and Galaxy, and (2) the agreement between observed and predicted slopes of the PL relations in the three galaxies. Distances to classical Cepheids, however, must be corrected for interstellar extinction, the amount of which can be much more sensitive to metal abundance than is the PL relation itself, if Cepheid colors are used to derive the reddening. Theoretical results are also presented for the bolometric PL relation, whose slope and dependence on metal abundance should be roughly applicable to infrared observations of Cepheids.

Distance moduli of the Magellanic Clouds, corrected for abundance effects, are $(m - M)_0 = 18.51 \pm 0.06$ (Large Cloud) and $(m - M)_0 = 18.80 \pm 0.06$ (Small Cloud).

Subject headings: galaxies: distances — galaxies: Magellanic Clouds — stars: abundances — stars: Cepheids — stars: pulsation

I. INTRODUCTION

The extragalactic distance scale still depends primarily on the period-luminosity (PL) and period-luminosity-color (PLC) relations for classical Cepheids. Historically, these relations have been assumed to be universal, but in recent years their dependences on chemical composition have come under increasing investigation because external galaxies are now known to exhibit a wide range of metal and helium abundances, not only among themselves but also within a given galaxy.

Reddish (1956) first approached the problem of theoretically predicting the PL relation by using simple homology relations (in the absence of detailed stellar models) together with an empirical period-radius relation. Thereby he was able to predict that Cepheids with more initial helium would be brighter. Later Sandage (1958) demonstrated explicitly that theory by itself predicts a PLC relation. Acting on this realization, Robertson (1973) used detailed stellar models computed by Stobie (1969), Cogan (1970), Iben and Tuggle (1972), and himself, together with observed properties of the Cepheid instability strip in the luminosity-effective temperature (LT_e) plane, and attempted to estimate abundance differences between Large Magellanic Cloud (LMC) Cepheids and Small Magellanic Cloud (SMC) Cepheids. Adopting the same theoretical models, Gascoigne (1974) derived a more accurate PLC relation, and, by introducing an observed PC relation, he semi-empirically obtained the PL relation. Considering only the metals abundance, he found that Cepheids in the metal-deficient SMC would be fainter than normal Galactic Cepheids with the same periods and colors by 0.1–0.2 mag, based on the PL relation, or by ~ 0.4 mag, based on the PLC relation.

Iben and Tuggle (1975) redid the calculation entirely theoretically, using new stellar models for an improved PLC relation, and found an explicit dependence of luminosity on both helium (Y) and metal (Z) abundances by mass. Adopting $Z = 0.005$ for the SMC and $Z = 0.02$ for the Galaxy together with a fixed helium abundance (as did Gascoigne), Iben and Tuggle predicted that the Cepheid luminosities would differ by ~ 0.3 mag, very much like Gascoigne's result. Unlike Gascoigne, however, they did not simply assume the values of Y and Z for the SMC, but rather solved for them by fitting theoretical blue edges to the observed blue edge in the LT_e and PL planes. They repeated this operation for the LMC ($Z = 0.01$) and for M31 ($Z = 0.02$), and found $Y = 0.29 \pm 0.06$ in all three galaxies. Elaborating on Iben and Tuggle's work, Iben and Renzini (1984) later introduced a theoretical ridge-line LT_e relation into the PLT_e relation, and so obtained a PL relation whose dependence on chemical composition turned out to be negligible.

Butler (1978) reapplied Iben and Tuggle's equations and methodology to improved data for SMC and LMC Cepheids. Not surprisingly, he obtained results that were essentially identical to theirs concerning the metal abundances. Using a similar approach, with theoretical data from Cox, King, and

Stellingwerf (1972) and Becker, Iben, and Tuggle (1977), Cogan (1980) independently reached the same conclusion. Pel, van Genderen, and Lub (1981) used Gascoigne's original approach, together with more recent theoretical data from Becker, Iben, and Tuggle (1977), and predicted that SMC Cepheids with $Z = 0.004$ would be fainter than Galactic Cepheids with $Z = 0.02$ by ~ 0.5 mag, based on the PLC relation. Van Genderen (1983) later turned the problem partly around: by adopting $(Y, Z) = (0.24, 0.004)$ for the SMC and $(Y, Z) = (0.25, 0.01)$ for the LMC, he was unable to detect observationally any obvious contradiction with Iben and Tuggle's theoretical PLC relation. Subsequently, Caldwell and Coulson (1986) adopted relative metal abundances corresponding to $Z = 0.02$, $Z = 0.014$, and $Z = 0.005$ for the Galaxy, LMC, and SMC, respectively, from Feast (1985), and assuming $\delta Y = 2.8 \delta Z$ from Lequeux *et al.* (1979), they then introduced these abundances into the theoretical PLC and PL relations of Iben *et al.* and thereby corrected their own strictly empirical PLC and PL relations. They made no attempt to check the theoretical results of Iben *et al.* empirically.

Certain drawbacks common to most of the modern investigations should be noted, however. First, very limited theoretical data have been used—essentially, only the stellar models of Iben *et al.* and of Cogan. No assessment of the dispersion of the theoretical quantities has been made—for example, by intercomparing the model results of many different authors. Second, in some of the constitutive relations among the variables, slopes that are made to depend on chemical composition have been adopted by the various users. Is this theoretically and observationally justified? Should not the composition dependence be placed wholly in the intercepts of the constitutive relations? Third, relative theoretical data and absolute theoretical data have been combined in most of these analyses. Yet, to take just one example, relative shifts of the blue edge are expected to be much more accurately predicted than are the absolute locations of the blue edges. Fourth, theoretical relations and observational relations have also been combined in some analyses. Theoretical and observational relations, however, possess very different sources of error, and their composition dependences cannot be consistently taken into account if they are combined. Fifth, in some studies, the composition dependence has been ignored in one or more of the theoretical relations. This has almost always been the case for the bolometric correction as a function of effective temperature or color. Sixth, the helium and metal abundances in Cepheids have usually been indirectly inferred and then used to *correct* the PLC and PL relations, instead of being derived from direct spectroscopic observations of Cepheids and related objects and used to *test* the PLC and PL relations.

To remedy these inadequacies, we have undertaken the present study. The derivation of the PLC and PL relations proceeds here entirely theoretically, using relevant published data from stellar evolution, atmosphere, and pulsation theory. Only relative data (no absolute data) are employed, and the composition dependences are provided for all quantities. The results then represent a true theoretical prediction. They are, as a consequence, mathematically self-consistent and amenable to error estimation, where all the uncertainties are theoretical ones.

Since some frequently used observational data (such as intrinsic colors and blue edges) still have significant uncertainties, it is preferable to avoid them for the purpose of the present work. In fact, the intrinsic $B-V$ color term in the PLC relation is so poorly known (Caldwell and Coulson 1986; Laney and Stobie 1986) that it cannot be used even to crudely check the theoretical predictions of this relation. On the other hand, the PL relation for normal Galactic Cepheids has been very accurately established by averaging a large number of determinations of it. The result is

$$M_V = -3.62 - 2.85(\log P - 0.8) ,$$
$$\pm 0.03 \quad \pm 0.03$$

where M_V is the time-averaged visual absolute magnitude, P is the period in units of days, and the errors quoted are standard errors of the mean, including both accidental and systematic errors (Stothers 1983; Carson and Stothers 1988). Owing to the close agreement recently found between theoretical and observed properties of normal Galactic Cepheids (Carson and Stothers 1988), the theoretical basis for the present work is believed to be securely established.

The need for a closer investigation of the composition dependence of the PL relation arises from the recent reduction of the observational error in the zero-point luminosity, which is now measured in hundredths of a magnitude. In contrast, the error was ± 1 mag 30 yr ago, and, even 10 yr ago, it amounted to several tenths of a magnitude.

II. PERIOD-LUMINOSITY AND PERIOD-LUMINOSITY-COLOR RELATIONS

Theoretical equations will be adopted here in the simplest form compatible with the accuracy of the data on which they are based. In practice, this means simple linear approximations. From stellar evolution theory we obtain the mass-luminosity relation for stars crossing the instability strip:

$$\log (L/L_\odot) = l_1 + l_2 \log (M/M_\odot) . \tag{1}$$

Pulsation theory connects mass, radius, and period of the fundamental mode:

$$\log P = p_1 + p_2 \log (M/M_\odot) + p_3 \log (R/R_\odot) . \tag{2}$$

Stellar atmosphere theory provides the following three relations between luminosity, radius, effective temperature, visual absolute magnitude, and bolometric correction:

$$\log (L/L_\odot) = 2 \log (R/R_\odot) + 4 \log (T_e/T_{e\odot}) , \tag{3}$$

$$M_V = 4.75 - 2.5 \log (L/L_\odot) - \mathrm{BC} , \tag{4}$$

$$\mathrm{BC} = b_1 + b_2 \log (T_e/T_{e\odot}) . \tag{5}$$

Equations (1)–(5) yield a period-luminosity-temperature relation. Rather than display this relation as such, we convert it to two other forms that are more useful here. One of these is the PL relation, obtained by introducing the theoretical ridge-line expression

$$\log (T_e/T_{e\odot}) = t_1 + t_2 \log (L/L_\odot) . \tag{6}$$

Consequently,

$$M_V(\mathrm{PL}) = A + B \log P , \tag{7}$$

where

$$A = 4.75 - p_1 B + l_1 B p_2/l_2 + t_1(2Bp_3 - b_2) - b_1 , \tag{8}$$

$$B = -(5 + 2t_2 b_2)[2p_2/l_2 + p_3(1 - 4t_2)]^{-1} . \tag{9}$$

The second relation is obtained by substituting intrinsic $B-V$ color for effective temperature, using the expression

$$\log (T_e/T_{e\odot}) = c_1 + c_2(B-V)_0 . \tag{10}$$

Then a PLC relation follows from equations (1)–(5) and (10) as

$$M_V(\mathrm{PLC}) = A' + B' \log P + C'(B-V)_0 , \qquad (11)$$

where

$$A' = 4.75 - p_1 B' + l_1 B' p_2 / l_2 + c_1(2B' p_3 - b_2) - b_1 , \quad (12)$$

$$B' = -5(2p_2/l_2 + p_3)^{-1} , \qquad (13)$$

$$C' = c_2(2B' p_3 - b_2) . \qquad (14)$$

Other magnitudes and colors may of course be used, but in this paper coefficients will be derived numerically only for the B, V system.

In theory, the slopes A, B, A', B', and C' are functions of chemical composition (and, strictly speaking, are not even constant with period). This fact makes both the PL and the PLC relations "statistical" relations. In addition, both relations depend on the use of an average mass-luminosity relation, which does not discriminate between stars in different crossings of the instability strip or stars with different rotational angular momenta, magnetic fields, or amounts of mass loss. Since the PL relation, however, strictly represents the ridge line, it is "more statistical" than the PLC relation.

III. EVALUATING THE SLOPES IN THE RELATIONS

In practice, the slopes of the constitutive relations (eqs. [1]–[6] and [10]) are found to be rather insensitive to chemical composition. Specifically, the total dispersion among the values derived for the slope of any particular relation by different investigators for effectively the same chemical composition usually exceeds the dispersion of the values of the slope derived by each investigator for a wide range of chemical compositions. Therefore, we will ignore composition dependences in the slopes but retain them for the intercepts, where they are significant. Since only the slopes determine B, B', and C', our basic assumption about their compositional invariance can be tested observationally; the outcome of such a test for B (§ XI) confirms that B is effectively independent of composition. The adopted values of the slopes are now summarized.

Pulsation theory yields $p_2 = -3/4$ and $p_3 = 7/4$, approximately, for short-period Cepheid models (Christy 1966; Fricke, Stobie, and Strittmatter 1971; Cox, King, and Stellingwerf 1972; Cogan 1978; Vemury and Stothers 1978) and $p_2 = -1$ and $p_3 = 2$, approximately, for long-period Cepheid models (Cogan 1970; Carson and Stothers 1984). Since the zero point of the Cepheid distance scale is conventionally set at $\log P = 0.8$ (de Vaucouleurs 1978), we adopt $p_2 = -3/4$ and $p_3 = 7/4$. Notice that these values of p_2 and p_3 lie exactly midway between the values that are applicable to long-period Cepheids and those that are valid for main-sequence stars for which the traditional Q value ($Q \propto PM^{1/2}R^{-3/2}$) is nearly a constant.

The slope l_2 of the theoretical mass-luminosity relation has been found to be 3.48, 3.50, 3.57, and 3.68 by King *et al.* (1973), Carson and Stothers (1984), Stobie (1969), and Becker, Iben, and Tuggle (1977), respectively. We adopt $l_2 = 3.6$.

For the slope of the fundamental blue edge of the theoretical instability strip in the LT_e plane, various authors have obtained $d \log T_e / d \log L = -0.036$ (Deupree 1980), -0.039 (Carson and Stothers 1976), -0.041 (Stobie 1969; Iben and Tuggle 1972), -0.042 (King *et al.* 1973), -0.046 (Iben and Tuggle 1975), -0.050 (Baker and Kippenhahn 1965, 7 $M_\odot$; Hofmeister 1967, 9 $M_\odot$), and -0.051 (King *et al.* 1975). Cox and Hodson (1978) got -0.04 for the mass range 4–9 $M_\odot$, but their steeper slope for higher masses is now known to be incorrect (Cox 1987). All these values were derived by the use of Los Alamos opacities in the stellar models. Carson and Stothers (1976) have used, in addition to the Los Alamos opacities, the newer Carson opacities, deriving a rather different slope of -0.012, but since this slope was influenced by an incorrect (Carson *et al.* 1984) metal contribution to the opacities, we shall ignore it here. In general, the slope shows little sensitivity to the assumed mass-luminosity relation, although the inclusion of convection in the stellar models produces a slightly less negative slope than in the case of purely radiative models. Placing the Cepheid ridge line parallel to the fundamental blue edge, we adopt $t_2 = -0.04$. The observed slope of the Cepheid blue edge, although not very well determined, agrees satisfactorily with the theoretically predicted slope (Iben and Tuggle 1975; King *et al.* 1975; Carson and Stothers 1976).

Effective temperatures along the ridge line can be theoretically estimated by equating the ridge line with the central line of the theoretical instability strip (Cogan 1978; Deupree 1980) or by matching the pulsational properties of individual full-amplitude Cepheid models with those of observed ridge-line Cepheids (Carson and Stothers 1988). Both methods yield similar effective temperatures, which agree well with those directly observed, $T_e = 5500$–6000 K (Pel 1985). Although the dependence of atmospheric quantities on surface gravity is not very strong, we will use $\log g = 1.5$–3.0.

Bolometric corrections change with effective temperature in the Cepheid range with a slope b_2 given as 2.2 (Flower 1977), 2.5 (Bell and Gustafsson 1978; VandenBerg and Bell 1985), 2.8 (Lub and Pel 1977; Buser and Kurucz 1978; Kurucz 1979), and 3.5 (Johnson 1966). We adopt $b_2 = 2.5$.

The corresponding relation between effective temperature and intrinsic $B-V$ color has a slope c_2 equal to -0.15 (Johnson 1966), -0.16 (Schmidt 1972), -0.175 (Oke 1961; Kraft 1961; Rodgers 1970; Parsons 1971; Bell and Parsons 1974), -0.20 (Bell and Gustafsson 1978; Cox 1979, Fig. 2), -0.23 (Böhm-Vitense 1972; Flower 1977), -0.24 (Teays and Schmidt 1987), or -0.25 (Buser and Kurucz 1978; Kurucz 1979; VandenBerg and Bell 1985). We assign $c_2 = -0.20 \pm 0.05$. Since the uncertainty of the slope c_2 exerts by far the largest influence on the PLC relation compared with the uncertainties in all the other slopes (Table 1), a full range of estimated error is explicitly attached to it and will be carried along in the subsequent calculations.

TABLE 1

EFFECT OF CHANGES IN INPUT QUANTITIES ON THE SLOPES OF THE DERIVED PL, PLC, PC, AND PR RELATIONS

Changed Quantities	B	B'	C'	$(B-B')/C'$	G
Standard case[a]	−2.98	−3.75	3.12	0.25	0.72
$p_2 = -1/2, p_3 = 3/2$	−3.28	−4.09	2.95	0.27	0.79
$p_2 = -1, p_3 = 2$	−2.72	−3.46	3.27	0.23	0.66
$l_2 = 3.5$	−3.00	−3.78	3.15	0.25	0.72
$l_2 = 3.7$	−2.96	−3.72	3.09	0.25	0.72
$t_2 = -0.03$	−3.14	−3.75	3.12	0.20	0.73
$t_2 = -0.05$	−2.82	−3.75	3.12	0.30	0.71
$c_2 = -0.15$	−2.98	−3.75	2.34	0.33	0.72
$c_2 = -0.25$	−2.98	−3.75	3.91	0.20	0.72
$b_2 = 2.2$	−3.00	−3.75	3.06	0.25	0.72
$b_2 = 2.8$	−2.96	−3.75	3.18	0.25	0.72

[a] Parameters: $p_2 = -3/4$, $p_3 = 7/4$, $l_2 = 3.6$, $t_2 = -0.04$, $c_2 = -0.20$, $b_2 = 2.5$.

IV. COMPOSITION SPREAD OF THE INTERCEPTS

Collecting results so far, we have

$$A = 4.75 + 2.98p_1 + 0.62l_1 - 12.91t_1 - b_1 , \quad (15)$$

$$A' = 4.75 + 3.75p_1 + 0.78l_1 - 15.62c_1 - b_1 , \quad (16)$$

$$B = -2.98 , \quad B' = -3.75 , \quad C' = 3.1 \pm 0.8 . \quad (17)$$

By assumption, chemical composition changes affect only the intercepts, and therefore only the zero points A and A'. The resulting changes in M_V are

$$\delta M_V(\mathrm{PL}) = 2.98\delta p_1 + 0.62\delta l_1 - 12.91\delta t_1 - \delta b_1 , \quad (18)$$

$$\delta M_V(\mathrm{PLC}) = 3.75\delta p_1 + 0.78\delta l_1 - 15.62\delta c_1 - \delta b_1 . \quad (19)$$

Abundance changes, even extreme ones, leave p_1 practically unaltered (Fricke, Stobie, and Strittmatter 1971; Iben and Tuggle 1972, 1975; King *et al.* 1975). To a good approximation,

$$\delta p_1 = 0 . \quad (20)$$

To evaluate δl_1 and δt_1, sequences of evolutionary models and pulsational models for short-period Cepheids must be examined on the H-R diagram. Published models for stars of 5 $M_\odot$ reveal that the theoretical instability strip lies close to the blue tip of the evolutionary loop that occurs during core helium burning (Hofmeister 1967; Iben and Tuggle 1972; Carson and Stothers 1976; Becker, Iben, and Tuggle 1977). This theoretical prediction is fully confirmed by the observed locations of short-period Cepheids in open-cluster H-R diagrams, both in our Galaxy (Carson and Stothers 1976; Harris 1976; Schmidt 1984) and in the Large Magellanic Cloud (Arp 1967; Robertson 1974; Harris and Deupree 1976). Furthermore, an observed rough equality between the number of Cepheid variables and the number of other blue-loop giants in these clusters confirms that most Cepheids are actually burning core helium.

Evolutionary models for 5 $M_\odot$ stars at the tip of the blue loop exhibit the following dependences on initial helium and metal abundances: $\delta l_1 = 4.0\delta Y - 11.9\delta Z$ (Hallgren and Cox 1970), $\delta l_1 = 3.8\delta Y - 10.4\delta Z$ (Robertson 1971), and $\delta l_1 = 3.1\delta Y - 14.4\delta Z$ (Becker, Iben, and Tuggle 1977). We therefore adopt

$$\delta l_1 = 3.5\delta Y - 12\delta Z . \quad (21)$$

It should be noted that, well away from the blue tip, $\delta l_1 = 3.2\delta Y - 20.2\delta Z$ (Robertson 1973) or $\delta l_1 = 4.0\delta Y - 35.8\delta Z$ (Becker, Iben, and Tuggle 1977). Such large values of the δZ coefficient would have a significant effect on δM_V, but are not thought to be applicable to most Cepheids.

For the fundamental blue edge calculated over the mass range 5–7 $M_\odot$, $d \log T_e/dY$ has been estimated as 0.07 (Deupree 1980), 0.11 (Iben and Tuggle 1975), 0.125 (Iben and Tuggle 1972), 0.14 (Cox *et al.* 1973), 0.16 (King *et al.* 1975), and 0.17 (Stobie 1969). Similarly, estimates of $d \log T_e/dZ$ are -0.51 (Stobie 1969), -0.55 (Iben and Tuggle 1972, 1975), and -0.56 (Deupree 1980). On the assumption that the Cepheid ridge line is parallel to the fundamental blue edge, we adopt

$$\delta t_1 = 0.13\delta Y - 0.55\delta Z . \quad (22)$$

The blue edges, however, were calculated for purely radiative envelopes. If classical Cepheids behave like RR Lyrae stars, the dependence of the blue edge on δY for envelopes with time-dependent convection included might be closer to $d \log T_e/dY = 0.06$ (Stellingwerf 1984; Deupree 1985). Nevertheless, this difference makes only a slight change in our results for the PL relation, and none at all for the PLC relation, and so can probably be ignored in comparison with other uncertainties.

For b_1 and c_1, we have from model atmosphere calculations that include metallic line blanketing $\delta b_1 = 2\delta Z$, $\delta c_1 = 1.6\delta Z$ (Bell and Gustafsson 1978; VandenBerg and Bell 1985), and $\delta b_1 = 3\delta Z$, $\delta c_1 = 1.6\delta Z$ (Lub and Pel 1977; Buser and Kurucz 1978; Kurucz 1979). Since the corresponding dependences on δY are negligible (Sonneborn, Kuzma, and Collins 1979; Böhm-Vitense 1979), we adopt

$$\delta b_1 = 2.5\delta Z , \quad \delta c_1 = 1.6\delta Z . \quad (23)$$

V. COMPOSITION DEPENDENCE OF THE ZERO POINTS

Inserting equations (20)–(23) into equations (18) and (19) yields

$$\delta M_V(\mathrm{PL}) = 0.5\delta Y - 2.8\delta Z , \quad (24)$$

$$\delta M_V(\mathrm{PLC}) = 2.7\delta Y - 36.9\delta Z . \quad (25)$$

Thus the PLC relation is far more sensitive to chemical abundance differences than is the PL relation, the main reason being the strong metal dependence of the conversion between intrinsic $B-V$ color and effective temperature (Table 2). However, even if the dependence were strictly zero, we would expect $\delta M_V(\mathrm{PLC}) = 2.7\delta Y - 11.9\delta Z$, which would still leave the PLC relation at least 5 times more sensitive to chemical abundance differences than is the PL relation.

The present PL relation displays a nonzero dependence of luminosity on metals abundance, in contrast to Iben and

TABLE 2

CONTRIBUTIONS OF INTERCEPTS OF CONSTITUTIVE RELATIONS TO TOTAL CHEMICAL COMPOSITION DEPENDENCE OF DERIVED RELATIONS δQ[a]

CONTRIBUTING INTERCEPT	δM_V(PL)		δM_V(PLC)		$\delta(B-V)_0^P$		$\delta \log (R/R_\odot)$	
	$\delta Q/\delta Y$	$\delta Q/\delta Z$	$\delta Q/\delta Y$	$\delta Q/\delta Z$	$\delta Q/\delta Y$	$\delta Q/\delta Z$	$\delta Q/\delta Y$	$\delta Q/\delta Z$
p_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
l_1	2.2	−7.4	2.7	−9.4	−0.2	0.6	−0.5	1.8
t_1	−1.7	7.1	...	...	−0.5	2.3	0.0	−0.1
c_1	...	...	0.0	−25.0	0.0	8.0	...	...
b_1	0.0	−2.5	0.0	−2.5	...	...	...	...
Total	0.5	−2.8	2.7	−36.9	−0.7	10.9	−0.5	1.7

[a] $\delta Q = \delta M_V(\mathrm{PL}), \delta M_V(\mathrm{PLC}), \delta(B-V)_0^P$, and $\delta \log (R/R_\odot)$, respectively.

Renzini's (1984) result. The main reason for this difference is that we have included the metal dependence of the bolometric correction. Similarly, our new PLC relation is considerably more sensitive to metals abundance than is Iben and Tuggle's (1975) result, primarily because those authors used a larger exponent in the mass-luminosity relation ($l_2 = 4.0$) and a weaker dependence of intrinsic $B-V$ color on metals abundance ($\delta c_1 = 1.3\delta Z$). In Iben and Tuggle's analysis, $\delta M_V(\mathrm{PLC}) \approx 2.2\delta Y - 21\delta Z$.

Notice that a metal-poor Cepheid is always fainter than a metal-rich Cepheid with the same period and color. (This is true despite the fact that at a fixed *mass* a metal-poor Cepheid is brighter.) Notice, too, that a helium-rich Cepheid is fainter than a Cepheid of normal helium, contrary to Reddish's (1956) result which was obtained before the advent of detailed stellar models.

VI. PERIOD-COLOR RELATION

From the PLC and PL relations it is possible to construct a ridge-line PC relation,

$$(B-V)_0 = [(A - A')/C' + [(B - B')/C'] \log P , \quad (26)$$

where the slope is, numerically, $(B - B')/C' = 0.25 \pm 0.06$. For variations of the chemical composition,

$$\delta(B-V)_0^P = -0.77\delta p_1 - 0.05\delta l_1 - 4.13\delta t_1 + 5.00\delta c_1 , \quad (27)$$

which leads to

$$\delta(B-V)_0^P = -0.7\delta Y + 10.9\delta Z . \quad (28)$$

This expression refers to the change in the "average" intrinsic $B-V$ color, at a fixed *period*, for a group of Cepheids with abundances that differ uniformly from the standard abundances. It is not to be confused with the change in intrinsic $B-V$ color of an individual Cepheid at a fixed *effective temperature*, which can be obtained from equation (10) and is

$$\delta(B-V)_0^T = -\delta c_1/c_2 = (8 \pm 2)\delta Z . \quad (29)$$

VII. TRANSFORMATION TO BLUE ABSOLUTE MAGNITUDE

To work out expressions in terms of δM_B instead of δM_V, it is necessary only to vary $M_B = M_V + (B-V)_0$ appropriately. Thus,

$$\delta M_B(\mathrm{PL}) = \delta M_V(\mathrm{PL}) + \delta(B-V)_0^P , \quad (30)$$

$$\delta M_B(\mathrm{PLC}) = \delta M_V(\mathrm{PLC}) + \delta(B-V)_0^T . \quad (31)$$

Since equation (26) for $(B-V)_0$ must be used in connection with the PL relation, the slope of the PL relation will be altered to $B + (B - B')/C'$. As for the PLC relation, the slope of the color term will be changed to $C' + 1$, but the slope of the period term will remain the same as before.

VIII. BOLOMETRIC RELATIONS

For completeness, we present "bolometric" versions of the two basic relations between period and luminosity. In the case of the PL relation, it is necessary only to set $b_1 = b_2 = 0$ in equations (8), (9), (15), and (18). For the PLT$_e$ relation, we introduce $b_1 = b_2 = 0$, $c_1 = 0$, and $c_2 = \log (T_e/T_{e\odot})/(B-V)_0$ into equations (12), (13), (14), (16), and (19). Then $B = -3.10$, $B' = -3.75$, $C' = 13.1$, and

$$\delta M_{\mathrm{bol}}(\mathrm{PL}) = 0.8\delta Y - 1.8\delta Z , \quad (32)$$

$$\delta M_{\mathrm{bol}}(\mathrm{PLT}_e) = 2.7\delta Y - 9.4\delta Z . \quad (33)$$

For comparison, Iben and Renzini (1984) obtained $B = -3.12$, $B' = -3.73$, $C' = 12.9$, $\delta M_{\mathrm{bol}}(\mathrm{PL}) = 0.0$, and $\delta M_{\mathrm{bol}}(\mathrm{PLT}_e) = 1.9\delta Y - 15.7\delta Z$. By treating Cepheids along the red edge of the instability strip as representative of the whole class, Robertson (1973) obtained $B = -2.87$ and $\delta M_{\mathrm{bol}}(\mathrm{PL}) = 1.2\delta Y - 6.6\delta Z$. These two strong dependences on δZ are believed to be physically unrealistic (§ IV), and so equations (32) and (33) are preferred here.

In the infrared, where bolometric corrections and intrinsic colors are less sensitive functions of temperature and metal abundance than in the visual region (McGonegal *et al.* 1982), the corresponding PL relation would be expected to have a slope and a composition dependence more closely resembling those predicted for the bolometric PL relation.

IX. INTERSTELLAR EXTINCTION AND TRUE DISTANCE MODULUS

To obtain the true distance modulus of a Cepheid with known period, the time-averaged V magnitude must first be corrected for interstellar extinction. Thus,

$$(m - M)_0 = (V - A_V) - M_V . \quad (34)$$

If A_V is obtained from observations of the Cepheid itself, we require knowledge both of the Cepheid's intrinsic $B-V$ color—so that the color excess $E_{B-V} = (B-V) - (B-V)_0$ can be computed—and of the ratio of total to selective extinction, $R_V = A_V/E_{B-V}$. Although both A_V and E_{B-V} depend on chemical composition, we assume that, in lowest order and with some observational justification (Caldwell and Coulson 1986), the slope R_V is invariant, and therefore only the composition dependence of $(B-V)_0$ need be considered. Hence

$$\delta(m - M)_0 = -\delta M_V + R_V\,\delta(B-V)_0 . \quad (35)$$

It follows, by using $\delta(B-V)_0^P$ in connection with the PL relation and $\delta(B-V)_0^T$ in connection with the PLC relation, that

$$\delta(m - M)_0(\mathrm{PL}) = -2.9\delta Y + 38.8\delta Z , \quad (36)$$

$$\delta(m - M)_0(\mathrm{PLC}) = -2.7\delta Y + (63 \pm 7)\delta Z , \quad (37)$$

where $R_V = 3.3$ has been adopted. Notice that the correction for change of intrinsic color in equation (35) must be applied regardless of the amount of interstellar reddening and extinction that are actually present.

If, on the other hand, A_V is found from colorimetric observations not of the Cepheid itself but of neighboring early-type stars or by some other method, the dependence of the true distance modulus on chemical composition of the observed Cepheid will be given by just

$$\delta(m - M)_0 = -\delta M_V . \quad (38)$$

There remains, of course, an implied dependence of A_V on chemical composition, inasmuch as the adopted intrinsic colors of the early-type stars will depend on composition. But in the case of such hot stars the corrections of $(B-V)_0$ and $(U-B)_0$ for atomic line blanketing (Buser and Kurucz 1978; Kurucz 1979) are very slight and can be ignored. Moreover, if A_V is obtained by fitting early-type stars in a cluster color-magnitude diagram, the $(B-V)_0$ color shift is only $\delta(B-V)_0 \approx -0.1\delta Y + 2.0\delta Z$ at fixed M_V, as can be deduced from theoretical models for unevolved main-sequence stars of 5–7 $M_\odot$ (Becker 1981).

X. PERIOD-RADIUS RELATION

An auxiliary expression that follows from equations (1), (2), (3), and (6) is the ridge-line period-radius relation,

$$\log (R/R_\odot) = F + G \log P , \quad (39)$$

where

$$F = -G[p_1 - l_1 p_2/l_2 + t_1(2p_2/l_2)(1-4t_2)^{-1}] , \tag{40}$$

$$G = [(2p_2/l_2)(1-4t_2)^{-1} + p_3]^{-1} . \tag{41}$$

Numerically, $G = 0.72$, and

$$\delta \log (R/R_\odot) = -0.5\delta Y + 1.7\delta Z . \tag{42}$$

The dependence of the period-radius relation on chemical composition is obviously very slight.

XI. DISCUSSION

Observationally, it is now known that Y lies between 0.23 and 0.30 (Stothers 1973; Lacy 1979; Wolff and Heasley 1985; Brown 1986; Peimbert 1986) and that Z ranges from very low values up to 0.02–0.04 (Nissen 1980; Bell and Dreiling 1981; Harris 1981; Harris and Pilachowski 1984). If Y and Z are uncorrelated, we have $|\delta M_V(\text{PL})| \leq 0.15$ and $|\delta M_V(\text{PLC})| \leq 1.66$. However, Y and Z appear to be loosely correlated, as a number of authors have determined from spectroscopic observations of H II regions in different galaxies. Reviewing all the evidence, Peimbert (1986) concludes that $\delta Y/\delta Z = 3.5 \pm 0.7$. In this case, $\delta M_V(\text{PL}) \approx -1.1\delta Z$ and $\delta M_V(\text{PLC}) \approx -27.4\delta Z$. Therefore, if $\delta Z = -0.015$, as is appropriate for a comparison of Cepheids in the solar neighborhood with those in a metal-poor galaxy like the SMC, we obtain $\delta M_V(\text{PL}) \approx 0.02$ and $\delta M_V(\text{PLC}) \approx 0.41$.

The derived results are not very sensitive to any reasonable changes made in the quantities entering the basic equations. For example, theoretical models of long-period Cepheids in the mass range 8–12 $M_\odot$ show $p_2 = -1$, $p_3 = 2$, $l_2 = 3.5$, $\delta l_1 = 2.5\delta Y - 11\delta Z$ (Carson and Stothers 1984) and $\delta p_1 = 0$, $t_2 = -0.04$, $\delta t_1 = 0.15\delta Y - 0.55\delta Z$ (Iben and Tuggie 1975). It follows that $B = -2.75$, $B' = -3.50$, $C' = 3.3 \pm 0.8$, $\delta M_V(\text{PL}) = -0.1\delta Y - 3.7\delta Z$, and $\delta M_V(\text{PLC}) = 2.5\delta Y - 39.9\delta Z$. Adopting $\delta Y/\delta Z \approx 3.5$, we obtain $\delta M_V(\text{PL}) \approx -4.0\delta Z$ and $\delta M_V(\text{PLC}) \approx -31.1\delta Z$. With $\delta Z = -0.015$, we finally get $\delta M_V(\text{PL}) \approx 0.06$ and $\delta M_V(\text{PLC}) \approx 0.47$. Again, the PLC relation leads to a composition-induced change of M_V that is 0.4 mag larger than that derived by using the PL relation.

Gascoigne (1974) and Iben and Renzini (1984) previously found almost the same differential effect, namely, ~0.3 mag. However, Gascoigne's PL result was $\delta M_V(\text{PL}) = 0.1$–$0.2$, which now seems too large, while Iben and Renzini obtained $\delta M_V(\text{PL}) = 0.0$, which seems too low. Van Genderen (1977) and Eggen (1985) found by purely observational means that $\delta M_V(\text{PL})$ increases with δZ—which is contrary to all the theoretical predictions. However, van Genderen (1983) appears now to have reversed himself, while Eggen's result depended on the validity of Sandage and Tammann's (1971) period-luminosity-amplitude relation, for which the use of amplitude as a proxy measure for color has been questioned by Butler (1978). In fact, two studies, one observational (Madore 1982) and the other theoretical (Carson and Stothers 1984), have concluded that amplitude *increases* with increasing color index, which is opposite in sense to what Sandage and Tammann (1971) and Cogan (1980) suggested.

The relative advantage gained by using the PL relation instead of the PLC relation is preserved even when the true distance modulus of a Cepheid is being computed. If the extinction correction is obtained from sources other than the Cepheid itself, there will be no change in the relative advantage because $\delta(m-M)_0 = -\delta M_V$. If the Cepheid's intrinsic $B-V$ color is used to derive the extinction correction, there is still no change, because $\delta(B-V)_0^P$ and $\delta(B-V)_0^T$ depend on δZ in almost the same way. Thus, if $\delta Y/\delta Z \approx 3.5$, we have $\delta(m-M)_0(\text{PL}) \approx 28.6\delta Z$ and $\delta(m-M)_0(\text{PLC}) \approx (54 \pm 7)\delta Z$. With $\delta Z = -0.015$, it follows that $\delta(m-M)_0(\text{PL}) \approx -0.43$ and $\delta(m-M)_0(\text{PLC}) \approx -0.81 \pm 0.10$.

Caldwell and Coulson (1986), in using Iben and Tuggle's (1975) theoretical formalism, obtained a somewhat weaker dependence of $\delta(m-M)_0(\text{PLC})$ on δZ than we have, and a negative dependence of $\delta(m-M)_0(\text{PL})$ on δZ. However, they merged theoretical and observational constitutive relations in getting their results, and used a very different approach in determining the reddening correction. Nevertheless, they too found the PL relation to be less sensitive than the PLC relation to the choice of chemical composition.

For these reasons, we prefer to adopt the PL relation. Despite its greater intrinsic scatter, which requires observations of more Cepheids to fix the mean line, the PL relation possesses a zero-point uncertainty of only a few hundredths of a magnitude arising from cosmic abundance variations. Although the remarkable weakness of this dependence on abundances is due to a near cancellation of the effect of *initial* envelope chemical composition on the star's inner structure (which determines the luminosity) and the effect of *present* envelope chemical composition on the star's outer structure (which determines the pulsational properties), the initial and present envelope abundances would have to differ by a much larger factor than is presently expected from normal stellar evolution theory (Iben 1964) for this near cancellation to be seriously affected. Furthermore, the potential effect of cosmic abundance variations is also limited by the observed correlation between Y and Z.

Some observational evidence exists to test the invariance, and hence universality, of the PL relation. In the Galaxy, LMC, and SMC, which have very different metallicities, the *slope* of the visual PL relation is found to be identical within the mean errors: -2.86 ± 0.05 (Galaxy, average of 11 determinations), -2.86 ± 0.05 (LMC, average of six determinations), and -2.84 ± 0.09 (SMC, average of three determinations), according to a compilation of published values of the slope made by Carson and Stothers (1988). In fact, the observed slope agrees remarkably well with the average value of the slope that we have predicted theoretically for Cepheids with both long and short periods: $B = -2.86 \pm 0.12$.

A second observational test makes use of the difference in visual absolute magnitude between classical Cepheids with log $P = 0.8$ and metal-poor RR Lyrae stars. This test assumes that metal-poor RR Lyrae stars, at least as a group, have the same visual absolute magnitude in all galaxies. The result of the test is $M_V(0.8) - M_V(\text{RR}) = -4.23 \pm 0.21$ (Galaxy), -4.16 ± 0.07 (LMC), and -4.19 ± 0.07 (SMC) (Stothers 1983). These two empirical checks on the slope and zero point of the PL relation essentially confirm the predicted universality of this relation in the relevant period range (3–100 days).

In principle, a more direct check of the invariance of the zero point could be effected by measuring statistical or secular parallaxes of Galactic Cepheids grouped according to chemical composition. Alternatively, radii for Cepheids with very different chemical compositions could be determined by the Baade-Wesselink method and its variants, and combined with effective temperatures to give composition-dependent luminosities. Unfortunately, the present accuracy of these two methods of deriving luminosities is too low in actual practice (see Table 1 in Stothers 1983) to apply to the question of composition invariance. Moreover, it would be necessary to look well outside the solar neighborhood to find Cepheids with sufficiently different compositions (Feast and Walker 1987). A third possibility is to identify groups of Cepheids with different chemical compositions in the same external galaxy.

An indirect check of the general theoretical basis of the PL relation can be made by comparing the predicted and observed slopes of the period-radius relation. Theory predicts $G = 0.72$ for short-period Cepheids and $G = 0.66$ for long-period Cepheids, the average value being $G = 0.69 \pm 0.03$. Observed Galactic Cepheids show $G = 0.680 \pm 0.015$ (the average of 23 determinations of G) (Carson and Stothers 1988). Unfortunately, the PLC relation serves this purpose poorly, since the predicted mean slope of the color term, $C' = 3.2 \pm 0.8$, has an uncertainty as large as that of the observed slope (Caldwell and Coulson 1986; Laney and Stobie 1986), while the predicted mean slope of the period term can be better checked by using the ordinary PL relation. The *bolometric* PL relation has a predicted mean slope of $B = -2.98 \pm 0.12$, but the observed slope of the infrared PL relation, which ought to be fairly close to the slope in the bolometric case, scatters between -2.90 and -3.48 (Welch and Madore 1984; Welch *et al.* 1985*a*, *b*; Fernley, Jameson, and Sherrington 1985; Mathewson, Ford, and Visvanathan 1986; Laney and Stobie 1986). Further observations are needed.

Adopting, therefore, the visual PL relation, we can refine our earlier calculation of the distance moduli of the Magellanic Clouds (Stothers 1983). Since the previous calculation used a zero-point luminosity based on Galactic Cepheids, together with interstellar extinction estimates derived from early-type stars, and took account of the mean positions of Cepheids within the Clouds through the use of mean apparent visual magnitudes at $\log P = 0.8$, the only presently needed adjustment to the distance moduli is a small shift of the adopted zero point to accommodate differences of chemical composition. Accordingly, with $Z = 0.02$ (Galaxy), $Z = 0.014$ (LMC), $Z = 0.005$ (SMC), and the assumption $\delta Y = 3.5\delta Z$, distance moduli for the Large and Small Clouds should be reduced by 0.01 and 0.02 mag, respectively. If, now, mean errors of ± 0.03 mag are assigned to (1) the zero point of the Galactic Cepheid visual absolute magnitudes, (2) the correction of this zero point for the different Cloud abundances, (3) the observed mean visual apparent magnitudes of Cloud Cepheids at $\log P = 0.8$, and (4) the adopted corrections for interstellar extinction (Stothers 1983), the revised moduli and their mean errors turn out to be

$$\text{LMC:} \quad (m - M)_0 = 18.51 \pm 0.06\,,$$

$$\text{SMC:} \quad (m - M)_0 = 18.80 \pm 0.06\,.$$

These results may be compared with distance moduli also obtained by using a PL relation for Cepheids but employing different observational and theoretical data and different approaches to the reddening correction. Incorporating adjustments for abundance differences, Caldwell and Coulson (1986) derived $(m - M)_0 = 18.55 \pm 0.06$ (LMC) and 19.03 ± 0.08 (SMC), while Feast and Walker (1987) found $(m - M)_0 = 18.47 \pm 0.15$ (LMC) and 18.78 ± 0.16 (SMC). Without making such adjustments, Laney and Stobie (1986) obtained 18.74 ± 0.06 (LMC) and 19.06 ± 0.06 (SMC), while Welch *et al.* (1987) found 18.57 ± 0.05 (LMC) and 18.93 ± 0.05 (SMC). All these authors used a zero-point luminosity determined exclusively by the method of Cepheid membership in a Galactic binary system, open cluster, or association. The zero-point luminosity used here, however, is based more securely on five different methods, is independent of the Hyades distance modulus, and represents a mean of 26 different determinations in which the five methods adopted give highly concordant results (Stothers 1983; Carson and Stothers 1988). Nevertheless, the rough agreement among various authors for the distance moduli of the Magellanic Clouds based on classical Cepheids is encouraging. The moduli found here are also fully consistent with those derived by using RR Lyrae stars (Stothers 1983) and related variables (Feast and Walker 1987), with $M_V(\text{RR}) = 0.6$.

It is worth emphasizing that the main problem in applying either the PL or PLC relation to derive distances to Cepheids is the sensitivity of the extinction correction to the metals abundance *if Cepheid colors are used to obtain the reddening.* Unfortunately, metal abundances remain difficult to calibrate in Cepheids, even on a differential basis (see, e.g., Luck and Lambert 1981). Observations using infrared magnitudes, which are advantageous for lessening the correction of the zero point for composition differences (§ VIII), also reduce the severity of the extinction problem and are ultimately to be preferred.

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RICHARD B. STOTHERS: Institute for Space Studies, NASA/Goddard Space Flight Center, 2880 Broadway, New York, NY 10025